# Gateway to A Level – GCSE revision

## B. Factorising quadratics

You need to be able to factorise quadratic expressions of the form $ax^2 + bx + c$.

If the coefficient of $x^2$ is $1$, look for a factorisation of the form $(x-p)(x-q)$. The numbers $p$ and $q$ are such that their product is $c$ and they add up to $b$.

### Worked example 1

Factorise $x^2 - 7x + 12$

| Solution | Comments |
|---|---|
| $x^2 - 7x + 12$ | Look at factors of $12$. You need two negative numbers to give $+12$.<br>$12 = (-1) \times (-12)$<br>$= (-2) \times (-6)$<br>$= (-3) \times (-4)$ |
| $= (x-3)(x-4)$ | $-3$ and $-4$ add up to $-7$. |

If the coefficient of $x^2$ is **not** $1\ x^2$, you need to adapt this procedure slightly. First look for two numbers that multiply to $ac$ and add up to $b$. Then split the middle term and factorise in pairs.

### Worked example 2

Factorise $6x^2 + 11x - 10$

| Solution | Comments |
|---|---|
| $6 \times (-10) = -60$ | You need two numbers that multiply to |

| | |
|---|---|
| | $-60$ and add up to $11$. This means you need one positive and one negative number, with the positive number being larger. |
| The two numbers are $-4$ and $15$<br>$(-4+15=11)$ | Look at factors of $60$:<br>$-60 = (-1)\times 60$<br>$= (-3)\times 20$<br>$= (-4)\times 15$ |
| $6x^2 + \mathbf{11x} - 10 = 6x^2 \mathbf{- 4x + 15x} - 10$ | Split the middle term: $11x = -4x + 15x$ |
| $= 2x(3x-2) + 5(3x-2)$ | Factorise in pairs: The first two terms have a common factor $2x$ and the last two terms have a common factor $5$. |
| $= (3x-2)(2x+5)$ | Finally, take out the common factor $(3x-2)$. |

An alternative method is to simply look for numbers that work. If the coefficients are small prime numbers this can be quite quick, but otherwise the method shown in Worked example 2 is more efficient.

## Worked example 3

Factorise $5x^2 + 9x - 2$

| Solution | Comments |
|---|---|
| $5x^2 + 9x - 2 = (5x)(x)$ | The only way to factorise $5$ is $5 \times 1$. |
| $= (5x-1)(x+2)$ | The missing numbers in brackets are $1$ and $2$.<br>One is positive one is negative.<br>Try possible combinations until you find one that gives the middle term $+9x$. |

Before you use one of these methods, you should check whether there is a common factor that can be taken out of all three terms. For example:

$$7x^2 - 35x + 42 = 7(x^2 - 5x + 6)$$
$$= 7(x - 2)(x - 3)$$

A special example of factorising a quadratic is the **difference of two squares**:

$$a^2 - b^2 = (a - b)(a + b)$$

## Worked example 4

Factorise $9x^2 - 25$

| **Solution** | **Comments** |
|---|---|
| $9x^2 - 25 = (3x - 5)(3x + 5)$ | $9x^2$ is the square of $3x$ and $25$ is the square of $5$. |

## EXERCISE B

Factorise these expressions:

**1** **a** $x^2 + 2x - 3$

**b** $x^2 - 2x - 35$

**c** $a^2 - 8a - 20$

**d** $b^2 + 13b + 40$

**2** **a** $2x^2 - 4x - 30$

**b** $5x^2 + 5x - 30$

**c** $2x^2 - 4x - 16$

**d** $3x^2 - 18x + 27$

**3** **a** $b^2 - 81$

**b** $k^2 - 100$

**c** $16y^2 - 49$

**d** $36x^2 - 81$

**4** **a** $3x^2 - 14x - 5$

**b** $2x^2 - x - 3$

**c** $5x^2 - 14x - 3$

**d** $2x^2 + x - 10$

**e** $6x^2 - 5x + 1$

**f** $15x^2 + 13x + 2$

**g** $6x^2 - x - 15$

**h** $10x^2 \quad + x - 21$

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## E. Types of numbers

There are many different types of number and it is very important that you know some of the labels applied to them:

$\mathbb{N}$ Natural numbers $\{\ldots 1,\ 2,\ 3,\ 4 \ldots\}$

$\mathbb{Z}$ Integers $\{\ldots -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3 \ldots\}$

$\mathbb{Q}$ Rational numbers $\left\{\ldots -\frac{3}{5},\ 0,\ 5.3,\ 6,\ \frac{43\,984}{865} \ldots\right\}$

$\mathbb{Q}$ Irrational numbers $\{\ldots -\sqrt{2},\ 1-\sqrt{3},\ \pi \ldots\}$

$\mathbb{R}$ Real numbers $\left\{\ldots -\sqrt{2},\ 0,\ \pi,\ 5.3,\ 6,\ \frac{43\,984}{865} \ldots\right\}$

The rational numbers are any numbers that can be written as a fraction of two integers.

Irrational numbers cannot be written as a fraction of two integers. The main irrational numbers that you know at the moment are surds and anything involving π.

The real numbers comprise all numbers you have met so far. If no other indication is given, assume that you are working with real numbers.

## EXERCISE E

**1** Tick the boxes to indicate which sets of numbers the numbers are in. The first one has been done for you.

| | **Number** | $\mathbb{N}$ | $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{R}$ |
|---|---|---|---|---|---|
| **a** | $0$ | ✓ | ✓ | ✓ | ✓ |
| **b** | $1$ | | | | |
| **c** | $\frac{1}{3}$ | | | | |
| **d** | $-6$ | | | | |
| **e** | $-1.5$ | | | | |
| **f** | $0.\dot{3}$ | | | | |
| **g** | $\sqrt{2}$ | | | | |
| **h** | $\sqrt{25}$ | | | | |
| **i** | $\pi + 2$ | | | | |

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## F. Functions

A function is a rule that assigns to each input value a unique output value. To evaluate a function at a particular value, you substitute the given value into the algebraic expression.

Although you will be allowed to use your calculator in examinations, it is important that you are confident about rules of arithmetic so that you can apply them to algebraic expressions.

### Worked example 1

If $f(x) = x^2 - 3x$, find

**a** $f(7)$

**b** $f(-2)$

**c** $f\left(\frac{1}{2}\right)$

| **Solution** | **Comments** |
|---|---|
| **a** $f(7) = 7^2 - 3 \times 7$ $= 49 - 21$ $= 28$ | Substitute $x = 7$ into $x^2 - 3x$. |
| **b** $f(-2) = (-2)^2 - 3 \times (-2)$ | Substitute $x = -2$ into $x^2 - 3x$. |
| $= 4 + 6$ $= 10$ | Be careful with negatives: $(-2)^2 = 4$ and $-3 \times (-2) = +6$ |
| **c** | Substitute $x = \frac{1}{2}$ into $x^2 - 3x$. |

| | |
|---|---|
| $\mathrm{f}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - 3 \times \frac{1}{2}$ | |
| $= \frac{1}{4} - \frac{3}{2}$<br>$= \frac{1}{4} - \frac{6}{4}$<br>$= -\frac{5}{4}$ | Make sure you are comfortable with the arithmetic of fractions. |

You also need to simplify functions when the input is itself a function of $x$.

## Worked example 2

If $\mathrm{f}(x) = x^2 - 3x$ find

**a** $\mathrm{f}(2x)$

**b** $\mathrm{f}(x+1)$

**c** $\mathrm{f}(-x)$

| **Solution** | **Comments** |
|---|---|
| **a**<br>$\mathrm{f}(2x) = (2x)^2 - 3 \times 2x$ | Substitute $2x$ into $x^2 - 3x$. |
| $= 4x^2 - 6x$ | Make sure you square **all** of $2x: (2x)^2 = 4x^2$ |
| **b**<br>$\mathrm{f}(x+1) = (x+1)^2 - 3 \times (x+1)$ | Substitute $x + 1$ into $x^2 - 3x$. |
| $= x^2 + 2x + 1 - 3x - 3$<br>$= x^2 - x - 2$ | Expand: $(x+1)^2 = (x+1)(x+1) = x^2 + 2x + 1$ |
| **c**<br>$\mathrm{f}(-x) = (-x)^2 - 3 \times (-x)$ | Substitute $-x$ into $x^2 - 3x$ |
| $= x^2 + 3x$ | Be careful with negatives: $(-x)^2 = x^2$ and $-3 \times (-x) = +3x$ |

1 Without using a calculator, evaluate each function at the given values.

**a** $\mathrm{f}(x) = (2 - 3x)^2$ at

**i** $\mathrm{f}(1)$

**ii** $\mathrm{f}(-4)$

**iii** $\mathrm{f}\left(\frac{1}{6}\right)$

**b** $\mathrm{g}(x) = x^3 - 2x + 1$ at

**i** $\mathrm{g}(3)$

**ii** $\mathrm{g}(-1)$

**iii** $\mathrm{g}(0)$

**c** $\mathrm{h}(x) = \sqrt{2x - 1}$ at

**i** $\mathrm{h}(5)$

**ii** $\mathrm{h}\left(\frac{1}{2}\right)$

**iii** $\mathrm{h}(13)$

**d** $\mathrm{f}(x) = \dfrac{1}{3 - 4x}$ at

**i** $\mathrm{f}(1)$

**ii** $\mathrm{f}(-3)$

**iii** $\mathrm{f}\left(-\frac{1}{4}\right)$

**e** $\mathrm{g}(x) = \dfrac{2}{\sqrt{5 - x^2}}$ at

**i** $\mathrm{g}(-1)$

**ii** $\mathrm{g}(2)$

**iii** $\mathrm{g}(-2)$

**f** $\mathrm{h}(x) = \dfrac{3x + 2}{x - 4}$ at

**i** $\mathrm{h}(0)$

**ii** $h\left(\frac{2}{3}\right)$

**iii** $h(6)$

Simplify each function for the given inputs.

$f(x) = 2x + 5$ at

$f(x-1)$

$f(x+2)$

$f(3x)$

$g(x) = 4 - 3x$ at

$g(-x)$

$g(x-2)$

$g\left(\frac{1}{3}x\right)$

$h(x) = x^2 + 2x - 3$ at

$h(2x)$

$h(x+3)$

$h(x-1)$

$f(x) = 4 - x - 2x^2$ at

$f(x-3)$

$f(-x)$

$f(-2x)$

$g(x) = \dfrac{1}{3-2x}$ at

$g(x+1)$

$g(x-1)$

$g\left(-\frac{1}{2}x\right)$

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## G. Rules of indices

You need to be able to evaluate positive, negative and fractional powers.

### Worked example 1

Evaluate:

**a** $(-4)^3$

**b** $\left(\frac{2}{3}\right)^{-2}$

**c** $64^{\frac{1}{3}}$

| **Solution** | **Comments** |
|---|---|
| **a** $(-4)\times(-4)\times(-4)$ | Write the exponent form as repeated multiplication. |
| $=16\times(-4)=-64$ | Evaluate in parts. |
| **b** $\left(\frac{2}{3}\right)^2=\left(\frac{2}{3}\right)\times\left(\frac{2}{3}\right)=\frac{4}{9}$ | $x^{-2}=\frac{1}{x^2}$, so evaluate $\left(\frac{2}{3}\right)^2$ first … |
| $\therefore\left(\frac{2}{3}\right)^{-2}=\frac{9}{4}$ | … and then take the reciprocal. |
| **c** $64^{\frac{1}{3}}=\sqrt[3]{64}=4$ | The power $\frac{1}{3}$ means cube root. |

You should also know the rules for working with powers:

$$x^a\times x^b=x^{a+b}\qquad \frac{x^a}{x^b}=x^{a-b}\qquad (x^a)^b=x^{ab}$$

## Worked example 2

Simplify:

**a** $\dfrac{x^3 \times x^5}{x^2}$

**b** $(x^3)^4 \div (x^5)^2$

| **Solution** | **Comments** |
|---|---|
| **a** $\dfrac{x^3 \times x^5}{x^2} = \dfrac{x^8}{x^2}$ | Add powers when multiplying. |
| $= x^6$ | Subtract powers when dividing. |
| **b** $(x^3)^4 \div (x^5)^2 = x^{12} \div x^{10}$ | Multiply powers. |
| $= x^2$ | Subtract powers when dividing. |

1 Are these statements true or false?

a $x^2$ is always larger than $x$

b $3 \times 5^7 = 15^7$

c $6^{14} = 6 \times 6^{13}$

d $1^{99} = 1$

e $0^{40} = 1$

f $9^7 = 7^9$

g $2^4 + 2^4 = 2^5$

h $3^4 + 3^4 = 3^5$

2 Evaluate without a calculator:

a $5 \times 2^3$

b $\left(\frac{2}{3}\right)^3$

c $\left(-\frac{1}{4}\right)^2$

d $\left(-\frac{5}{3}\right)^3$

3 Evaluate the following without a calculator, leaving your answers as a fraction where appropriate:

a i $4^{\frac{1}{2}}$

ii $8^{\frac{1}{3}}$

b i $10000^{0.5}$

ii $81^{0.25}$

c i $\left(\frac{1}{25}\right)^{\frac{1}{2}}$

ii $\left(\frac{9}{16}\right)^{\frac{1}{2}}$

d ii $8^{\frac{2}{3}}$

$25^{\frac{3}{2}}$

$100^{2.5}$

$81^{0.75}$

$\left(\frac{1}{16}\right)^{\frac{5}{4}}$

$\left(\frac{8}{27}\right)^{\frac{5}{3}}$

$8^{-\frac{1}{3}}$

$49^{-\frac{1}{2}}$

$\left(\frac{16}{9}\right)^{-\frac{1}{2}}$

$\left(\frac{9}{16}\right)^{-\frac{3}{2}}$

Simplify:

$x^3 \times x^5 \times x^2$

$\frac{x^7}{x^2 \times x^3}$

$\frac{(x^2)^3 \times x^5}{x^4}$

$\frac{x^7}{(x^5)^2 \times x^3}$

Write $64\sqrt{2}$ in the form $2^n$.

Write $\sqrt[3]{81}$ in the form $3^n$.

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## H. Surds

A surd (also called a root or a radical) is any number of the form $a + b\sqrt{c}$ where $a$, $b$ and $c$ are fractions or whole numbers.

The most important rule of surds deals with their product:

$\sqrt{ab} = \sqrt{a}\sqrt{b}$

There is no equivalent for the sum. In general:

$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

You can collect square roots of the same number together, so, for example:

$3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$.

One very useful tool for simplifying square roots is to take out any square factors of the number being square rooted. For example:

$\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$.

### Worked example

Write $\left(\sqrt{6} + \sqrt{2}\right)^2$ in the form $a + b\sqrt{3}$.

| **Solution** | **Comments** |
|---|---|
| $\left(\sqrt{6} + \sqrt{2}\right)\left(\sqrt{6} + \sqrt{2}\right)$<br><br>$= 6 + \sqrt{6}\sqrt{2} + \sqrt{2}\sqrt{6} + 2$ | Treat the expression as two brackets. |
| $= 8 + 2\sqrt{12}$ | Simplify. |

| | |
|---|---|
| $= 8 + 2\sqrt{4}\sqrt{3}$<br>$= 8 + 4\sqrt{3}$ | Then put it into the required form. |

**EXERCISE H**

**1** Write in the form $k\sqrt{5}$, where $k$ is a whole number:

**a** $7\sqrt{5} - 2\sqrt{5}$

**b** $\sqrt{5} + 9\sqrt{5} - 3\sqrt{5}$

**c** $\sqrt{20} + \sqrt{45}$

**d** $\sqrt{5} - 2(\sqrt{125} - \sqrt{20})$

**e** $3\sqrt{80} - 5\sqrt{20}$

**f** $\sqrt{80} + 7\sqrt{5}$

**2** Write in the form $\sqrt{a}$ where $a$ is a whole number:

**a** $4\sqrt{2}$

**b** $7\sqrt{3}$

**c** $\sqrt{7} + 2\sqrt{7}$

**d** $\sqrt{3} + \sqrt{75}$

**e** $\sqrt{2} + \sqrt{8}$

**f** $\sqrt{32} + \sqrt{8}$

**3** Write in the form $a + b\sqrt{3}$ :

**a** $2(3 - \sqrt{3}) - 3(1 - \sqrt{3})$

**b** $(1 + \sqrt{3}) - (1 - \sqrt{3})$

**c** $(1 + \sqrt{3})(2 + \sqrt{3})$

**d** $(1 - \sqrt{3})(1 + 2\sqrt{3})$

**e** $(4 - 2\sqrt{3})^2$

**f** $(\sqrt{15} - \sqrt{5})^2$

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## K. Direct and inverse proportion

Proportion is another word for fraction – it is a way of comparing one group to the whole quantity. So the proportion of vowels in the alphabet is $\frac{5}{26}$.

If two variables are 'in proportion' then if you divide one by the other the result is always the same. i.e. if $y$ is proportional to $x$ then $\frac{y}{x} = k,$ a constant. You usually just rewrite this as $y = kx$. (Sometimes you use the phrase 'direct proportion' for this type of relationship.)

### Worked example 1

If $a$ is proportional to $b$, and when $a = 6, b = 10$, what is the value of $a$ when $b = 34$?

| Solution | Comments |
|---|---|
| If $a$ is proportional to $b$ then $a = kb$. | |
| When $a = 6, b = 10$ so:<br>$6 = 10k$<br>$k = 0.6$ | Use the given values to find $k$. |
| So $a = 0.6 \times 34 = 20.4$ | |

There are other types of proportional relationships. For example:

- $y$ is proportional to $x^2$ means that $y = kx^2$
- $y$ is inversely proportional to $x$ means that $y = \frac{k}{x}$.

### Worked example 2

Given that $y$ is inversely proportional to $x^2$, and that $y = 40$ when $x = 2$, find an expression for $y$ in terms of $x$.

| **Solution** | **Comments** |
|---|---|
| $y = \dfrac{k}{x^2}$ | |
| $40 = \dfrac{k}{2^2} \Rightarrow k = 40 \times 4 = 160$ | Use the given information to find $k$. |
| $\therefore y = \dfrac{160}{x^2}$ | |

## EXERCISE K

**1** **a** If $a$ is proportional to $b$ and when $\mathrm{a} = 3, \mathrm{b} = 6,$ find a formula for $a$ in terms of $b$.

**b** If $r$ is proportional to $s$ and when $r = 20,\ s = 15,$ find a formula for $r$ in terms of $s$.

**c** If $x$ is proportional to $y$ and when $x = 4, y = 10$, find $x$ when $\mathrm{y} = 12$.

**d** If $x$ is proportional to $y$ and when $x = 10, y = 7$, find y when $x = 25$.

**e** If $p$ is proportional to $q^2$ and when $p = 5, q = 10,$ find a formula for $p$ in terms of $q$.

**f** If $r$ is proportional to $\sqrt{s}$ and when $r = 1, s = 4$, find $r$ when $s = 16$.

**2** **a** Given that $y$ is inversely proportional to $x$, and that $y = 4$ when $x = 3$, find the value of $y$ when $x = 2$.

**b** $F$ is inversely proportional to $r^2$ and $F = 35$ when $r = 2$. Find an expression for $F$ in terms of $r$.

**c** Given that $a$ is inversely proportional to $\sqrt{S},$ and that $a = 3$ when $S = 16$, find the value of $S$ when $a = 6$.

**d** If $c$ is inversely proportional to $d$, and $c = 12$ when $d = 3$, find an expression for $c$ in terms of $d$.

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## L. Using coordinates

You can describe any point in a plane relative to an origin using two numbers, which you call *coordinates*. The $x$-coordinate describes how far to the right the point is from the origin and the $y$ -coordinate describes how far up a point is from the origin. A negative coordinate means the point is to the left or down respectively.

You can calculate a gradient between two points, giving a measure of the steepness of the line connecting the two points. If the gradient is positive it *increases* from left to right. If it is negative it *decreases* from left to right. If the two points have coordinates $(x_1, y_1)$ and $(x_2, y_2)$ then:

**Key point**

Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$

There are also simple formulae for the point exactly half-way between two coordinates and the distance between two points.

**Key point**

Midpoint $= \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

**Tip**

Notice that this is just the average of the coordinates of the two points.

**Key point**

Distance between two points $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

**Tip**

This formula follows from Pythagoras' theorem.

## Worked example

Point $A$ has coordinates $(1,5)$ Point $B$ has coordinates $(-3,3)$

**a** What is the gradient of the line connecting $A$ and $B$?

**b** What are the coordinates of the midpoint of $A$ and $B$?

**c** What is the distance between $A$ and $B$?

| **Solution** | **Comments** |
|---|---|
| **a** $\text{Gradient} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-5}{-3-1} = -\dfrac{2}{-4} = \dfrac{1}{2}$ | Use the formula for gradient. |
| **b** $\text{Midpoint} = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right) = \left(\dfrac{1 + -3}{2}, \dfrac{5+3}{2}\right) = \left(\dfrac{-2}{2}, \dfrac{8}{2}\right) = (-1, 4)$ | Use the formula for midpoint. |
| **c** $\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-3-1)^2 + (3-5)^2} = \sqrt{(-4)^2 + (-2)^2} = \sqrt{16+4} = \sqrt{20}$ | Use the formula for distance. |

## EXERCISE L

**1** Find the gradient of the line connecting:

**a** $(1,4)$ and $(3,7)$

**b** $(4,-2)$ and $(9,2)$

**c** $(-3,-3)$ and $(5,-3)$

**2** Find the midpoint of these points:

**a** $(1,4)$ and $(3,7)$

**b** $(4,-2)$ and $(9,2)$

**c** $(-3,-3)$ and $(5,-3)$

**3** Find the distance between the two points:

**a** $(1,4)$ and $(3,7)$

**b** $(4,-2)$ and $(9,2)$

**c** $(-3,-3)$ and $(5,-3)$

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## M. Straight line graphs

If you are given a rule connecting the $x$-coordinate and the $y$-coordinate, only some points on the plane will satisfy them. If the rule is of the form $y = mx + c$ then those points will lie in a straight line with gradient $m$ and the line will meet the $y$-axis at $(0, c)$.

If you are given the gradient and one point the line passes through, you can use that information to find the value of $c$.

If two lines have the same gradient then they are parallel.

### Worked example

A line has equation $3x + 2y = 2$. Find the equation of a line parallel to this line through the point $(1, 2)$.

| **Solution** | **Comments** |
|---|---|
| $2y = 2 - 3x \Rightarrow \quad y = 1 - \frac{3}{2}x$<br>So the gradient is $-\frac{3}{2}$ | Rearrange into the form $y = mx + c$ to find the gradient. |
| The required line also has gradient $-\frac{3}{2}$ and passes through the point $(1, 2)$ | Parallel lines have the same gradient. |
| $y = mx + c$ :<br>$2 = -\frac{3}{2} \times 1 + c$<br>$\Rightarrow c = \frac{7}{2}$ | You know $m$ and one pair of ($x$, $y$) values. |
| $\therefore y = -\frac{3}{2}x + \frac{7}{2}$ | |

## EXERCISE M

**1** Find the gradient and $y$-intercept of these straight lines:

**a** $y = 2x - 7$

**b** $y = 5 - x$

**c** $3x + y = 9$

**d** $y + 5 = 3(x - 1)$

**e** $5y = 10x + 1$

**f** $3y + 2x = 0$

**2** Find the equation of a line:

**a** through $(1,\ 3)$ with gradient $7$

**b** through $(2,\ 2)$ with gradient $-\frac{1}{2}$

**c** through $(0,\ 0)$ with gradient $-5$.

**3** Find in the form $y = mx + c$ the equation of:

**a** a line parallel to $y = 3x - 1$ through $(1,\ 1)$

**b** a line parallel to $x + y = 5$ through $(2, 4)$

**c** a line parallel to $3x - 2y + 4 = 0$ through $(0,\ 0)$.

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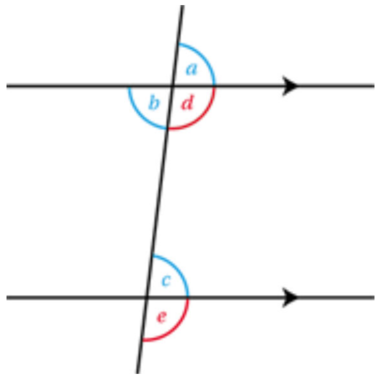
## N. Angles and bearings

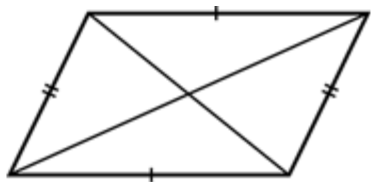
### Parallel lines




- $a = b = c$
- $d = e = 180° - a$

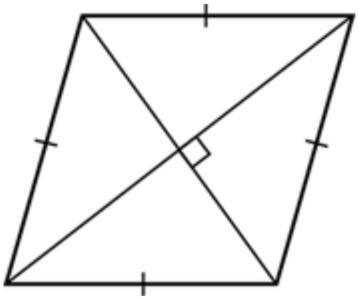
### Parallelograms and rhombuses

Parallelogram:



- opposite sides are equal
- opposite angles are equal
- adjacent angles add up to $180°$
- diagonals bisect each other

Rhombus:

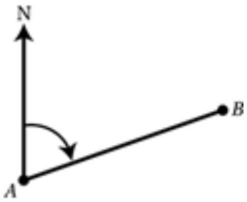


A rhombus is a type of parallelogram in which all four sides are equal. There are two further useful facts about a rhombus:

- the diagonals are perpendicular to each other
- the area is equal to $\frac{d_1 d_2}{2}$, where $d_1$ and $d_2$ are the lengths of the diagonals.

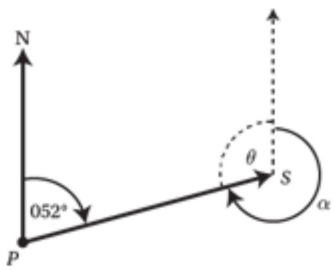
## Bearings

Bearings are a way of describing directions by specifying the angle measured clockwise from the north.



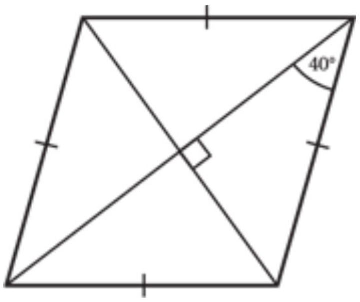

## Worked example

The bearing of $S$ from $P$ is $052°$ Find the bearing of $P$ from $S$.

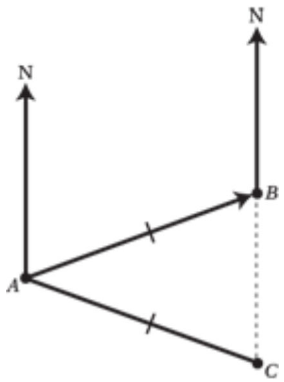
| **Solution** | **Comments** |
|---|---|
|   | Draw a diagram and label required angles. |
| $\theta = 180 - 52 = 128°$<br>$\therefore \quad \alpha = 360 - \theta = 232°$ | Angles $52°$ and $\theta$ add up to $180°$ (see 'parallel lines'). |

## EXERCISE N

**1** Find the angles at the four vertices of this rhombus:




**2** The bearing of $B$ from $A$ is $040°$. The distance from $A$ to $C$ is the same as the distance from $A$ to $B$, and $C$ is directly south of $B$.




Find the bearing of:

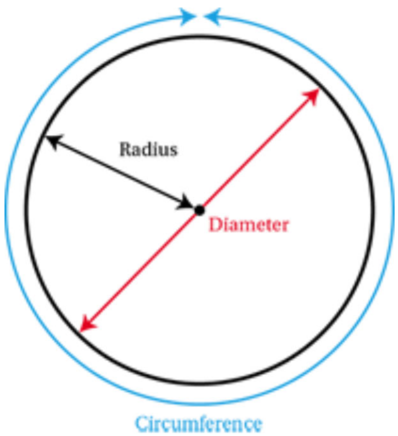
**a** $A$ from $B$

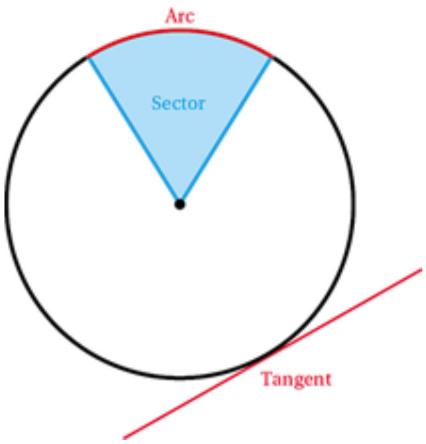
**b** $A$ from $C$.

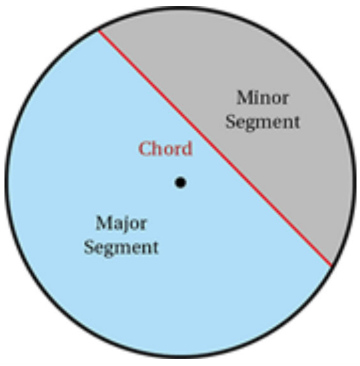
# Gateway to A Level – GCSE revision

## O. Circle theorems

You need to know this terminology:

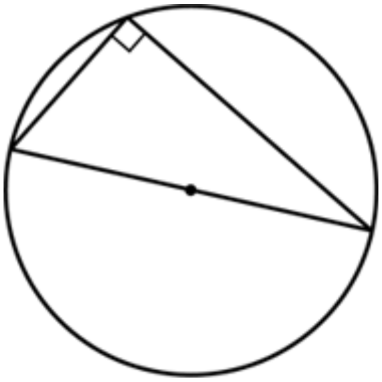


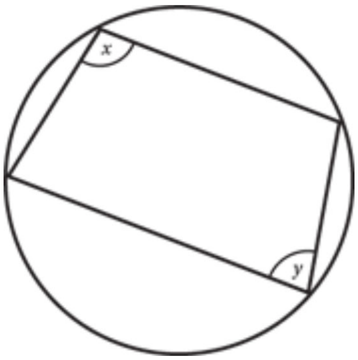




These theorems about angles in circles can be useful in solving problems about diameters and tangents:

**1** The angle in a semicircle is $90°$.

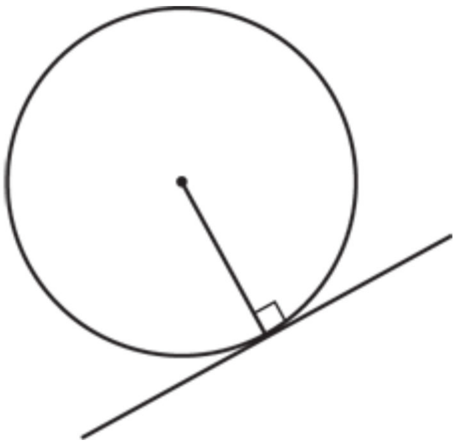


**2** Opposite angles in a cyclic quadrilateral (a quadrilateral with all four corners on the circumference of a circle) add up to $180°$.




$x + y = 180°$

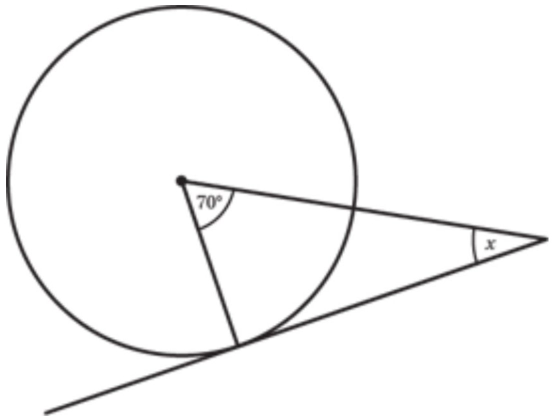
**3** A tangent meets its radius at right angles.



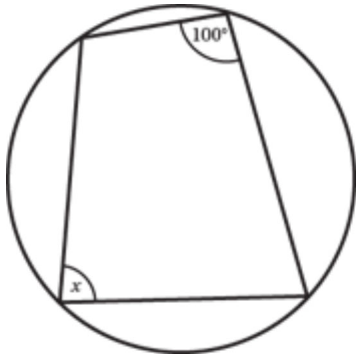
**EXERCISE O**

**1** Find the value of the angle marked $x$ in these diagrams, giving reasons for your answers:

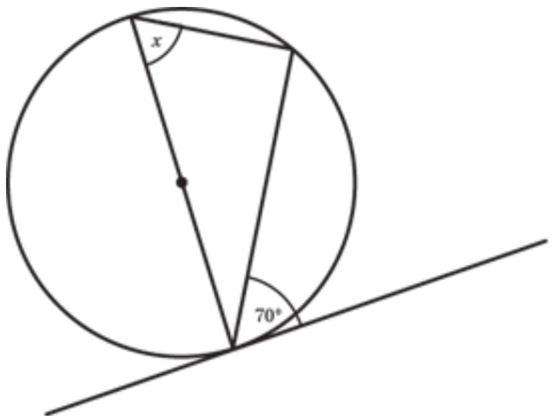
**a**




**b**




**c**



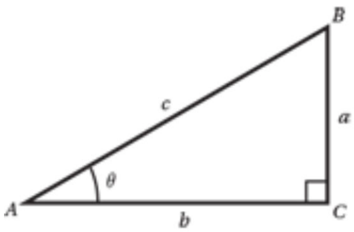
# Gateway to A Level – GCSE revision

## P. Pythagoras and trigonometry

There is a convention for labelling triangles with the angles in capital letters opposite the side with the same letter but in lower case.

In a right-angled triangle the longest side is always opposite the right angle and it is called the hypotenuse. These rules *only* apply to right-angled triangles:

**Key point**




Pythagoras' theorem:

$$c^2 = a^2 + b^2$$

Trigonometric ratios:

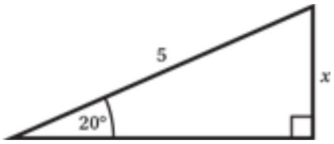
$$\sin\theta = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \cos\theta = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \tan\theta = \frac{a}{b} = \frac{\text{oppos}}{\text{adjac}}$$

Pythagoras' theorem also works in reverse – if you have a triangle with $c^2 = a^2 + b^2$ then it is right-angled with the right angle at $C$.

If you know the lengths in a right-angled triangle you can then use these to find the angles. To do this you have to 'undo' one of the trigonometric ratios to get just $\theta$ using the $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ buttons on your calculator.

## Worked example 1

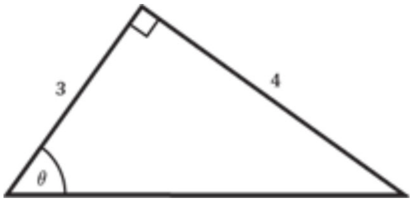
Find the length $x$ in this diagram:




| Solution | Comments |
|---|---|
| $\sin 20° = \dfrac{x}{5}$<br>$x = 5\sin 20° = 1.71(3\text{s.f.})$ | First decide which trig ratio to use. The relevant sides are the side opposite the angle and the hypotenuse – this means sin. |

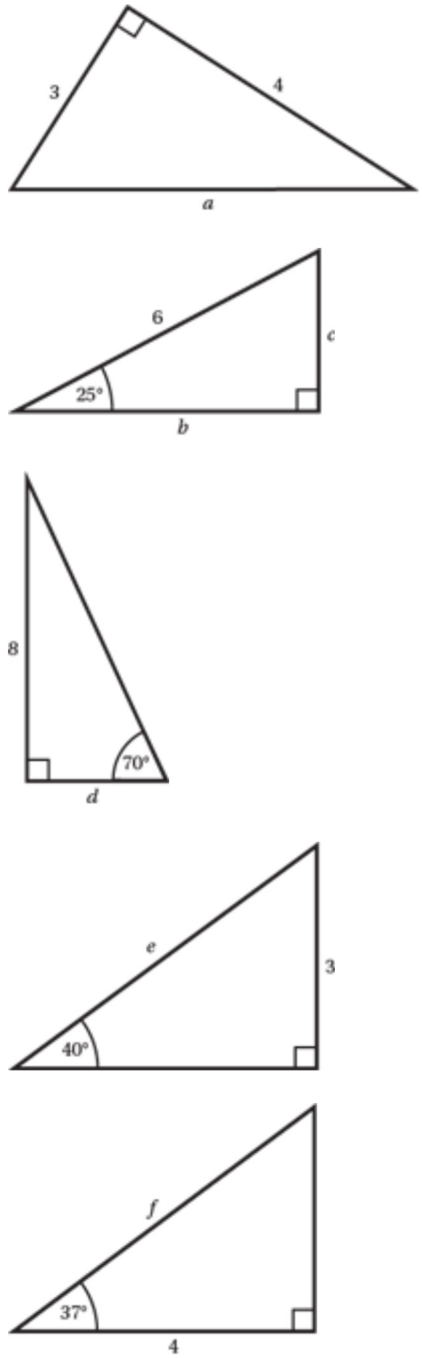
## Worked example 2

Find the angle $\theta$ in this diagram:

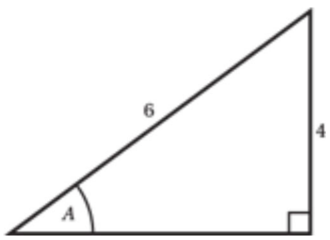


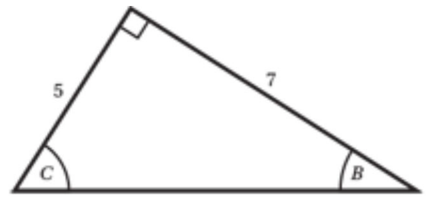

| Solution | Comments |
|---|---|
| $\tan\theta = \dfrac{4}{3}$ | First decide which trig ratio to use. The relevant sides are the side opposite the angle and the side adjacent to the angle – this means tan. |
| $\theta = \tan^{-1}\left(\dfrac{4}{3}\right) = 53.1°\left(3\text{ s.f.}\right)$ | To find $\theta$ you need to undo the tangent operation. |

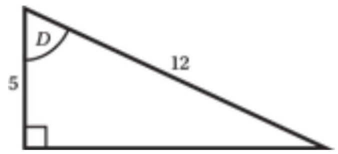
**3** Find the unknown lengths $a$ to $f$ :

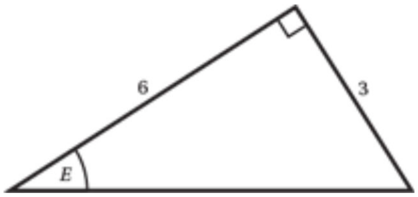


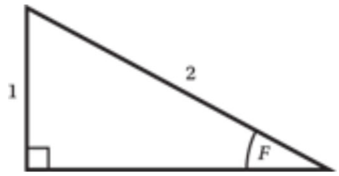

**2** Find the unknown angles $A$ to $F$:





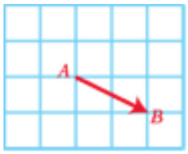

A ship is $10\,\mathrm{km}$ east and $8\,\mathrm{km}$ north of a lighthouse. Find the bearing of the ship from the lighthouse.

# Gateway to A Level – GCSE revision

## Q. Vectors

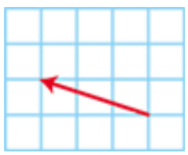
You can represent vectors either by drawing arrows or by using the column vector notation.

### Worked example 1




**a** Write the vector $\overrightarrow{AB}$ as a column vector.

**b** Represent the vector $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ on the grid.

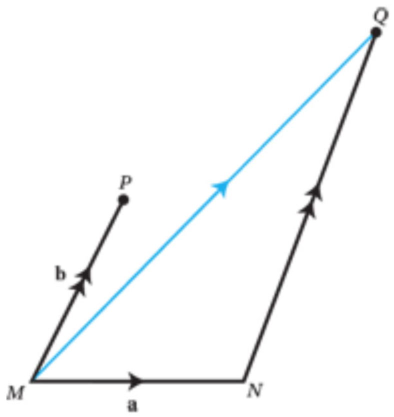
| **Solution** | **Comments** |
|---|---|
| $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ | The top number shows the number of units to the right.<br>The bottom number shows the number of units up. The vector goes 1 unit down, so it's a *negative* number. |
|  | The top number is *negative*, so the vector goes to the left and up. |

You can add vectors by joining the endpoint of one vector to the start of another. You can use this to express vectors in terms of other vectors.

### Worked example 2

The line $NQ$ is parallel to $MP$ and $NQ = 2MP$.

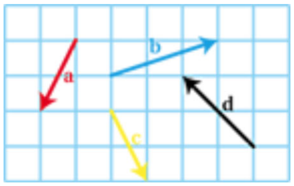
Express the vector $\overrightarrow{MQ}$ in terms of vectors $\mathbf{a}$ and $\mathbf{b}$.



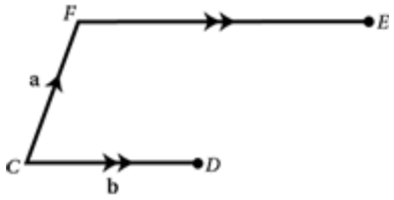

| Solution | Comments |
|---|---|
| $\overrightarrow{NQ} = 2\mathbf{b}$ | Going from $N$ to $Q$ is in the same direction but twice the distance as from $M$ to $P$. |
| $\overrightarrow{MQ} = \overrightarrow{MN} + \overrightarrow{NQ}$<br>$= \mathbf{a} + 2\mathbf{b}$ | You can get from $M$ to $Q$ by going from $M$ to $N$ and then from $N$ to $Q$. |

## EXERCISE Q

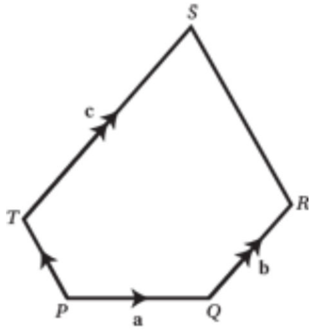
**1** Write these as column vectors:




**2** **a** $FE$ is parallel to $CD$ and $FE = 2CD$. Express $\overrightarrow{CE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.




**b** $RS$ is parallel to $PT$ and $RS = 2PT$. $TS$ is parallel to $QR$ and $TS = 2QR$.

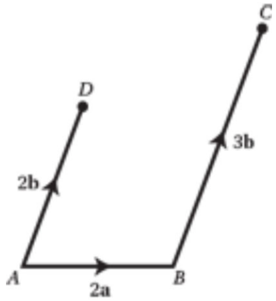



Express in terms of $\mathbf{a}$ and $\mathbf{b}$:

**i** $\overrightarrow{RP}$

**ii** **c**

**c** Express $\overrightarrow{DC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.



# Gateway to A Level – GCSE revision

## W. Tree diagrams

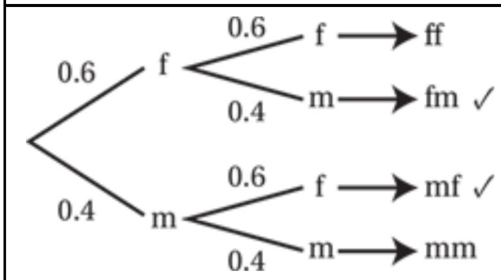
Listing all possible outcomes only helps calculate probabilities when the outcomes are equally likely. When this is not the case, tree diagrams can help find the probabilities instead.

The rule for calculating probabilities from a tree diagram is:

- multiply along branches
- add between branches.

### Worked example 1

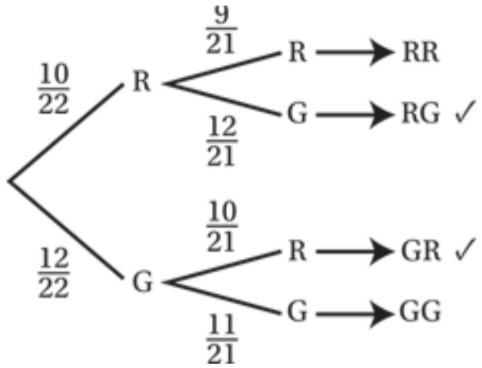
In a particular species of animal, the probability of a female offspring is $0.6$. A female has two offspring. Find the probability that there is a male and a female.

| Solution | Comments |
|---|---|
| | There are two possible outcomes, but they are not equally likely. So listing all possible combinations in a table wouldn't give the correct answer. |
|   | In a tree diagram, the first set of branches represents possible outcomes for the first offspring, and the second set the possible outcomes for the second offspring. |
| $(0.6 \times 0.4) + (0.4 \times 0.6) = 0.48$ | There are two ways to get a male and a female offspring. Multiply along branches and add between branches. |

The probabilities on the two sets of branches don't have to be the same; the probabilities can change depending on the outcome of the first event.

## Worked example 2

A bag contains $10$ red balls and $12$ green balls. Two balls are chosen at random and not replaced. Find the probability that the two balls are different colours.

| **Solution** | **Comments** |
|---|---|
|  | The first ball is chosen out of $22$.<br><br>The second ball is chosen out of $21$; but the probability of it being red depends on the colour of the first ball. |
| $\left(\frac{10}{22}\times\frac{12}{21}\right)+\left(\frac{12}{22}\times\frac{10}{21}\right)=0.519$ | There are two ways to get one red and one green ball. |

### EXERCISE W

**1** The probability that Daniel is late for school on any given day is $0.02$. Find the probability that he is late on exactly one out of two consecutive days.

**2** A box contains $6$ triangular tokens and $10$ square tokens.

**a** Two tokens are picked at random and not replaced. Find the probability that they are both square tokens.

**b** How does the answer change if the first token is replaced in the box?

**3** The probability that it is windy on any given day is $0.3$. If it is windy, Rosie walks to work with probability $0.8$ and cycles with probability $0.2$. If it is not windy, Rosie cycles with probability $0.9$ and walks with probability $0.1$. What is the probability that Rosie cycles to work on a randomly selected day?